MATH 280
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On the Envelopes of Parameterized Families of Curves

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Introduction

An envelope of a family of curves is a curve tangent to all members of this family at some point. We will provide a basis for the envelope algorithm, which can define an enveloping curve for any parameterized family of curves for which an envelope exists. We will also show the utility of envelopes and the envelope algorithm in solving the Ladder Problem, as well as providing a more simple geometric proof. Lastly, we will use envelopes to explore the geometry of the work of artist Emma Kunz.

Finding the Envelope

Although there exist many definitions for a mathematical envelope, perhaps the most intuitive in our context is the definition of the geometric envelope.

Definition. *Let $\mathcal{F}$ be a family of curves such that each $C_t \in \mathcal{F}$ is given by $F(x, y, t) = 0$ for some fixed t, where C_t is smooth, and parameter t lies in an open interval. Then, the* ***geometric envelope***[1] *of $\mathcal{F}$ is a smooth curve tangent at all points to some $C_t \in \mathcal{F}$.*

Importantly, envelopes are not necessarily unique, and some are larger than others. In fact, the geometric envelope is a subset of the discriminant envelope,[1] the envelope defined by the envelope algorithm. Before proving this algorithm, first, it is important to develop intuition on how one might derive such an algorithm. Since an envelope must be tangent at all points to some $C_t \in \mathcal{F}$, all points contained in the envelope must also be contained in some C_t. Therefore, all points on an enveloping curve E must satisfy $F(x, y, t) = 0$.

The second property which enables the formation of the envelope algorithm may be less obvious. As two nearby curves approach one another, the limit of their intersection approaches a point on the enveloping curve.[2] Another subset of the discriminant envelope, the limiting position envelope,[1] is defined this way. Let C_t and $C_{t'}$ be nearby curves in family $\mathcal{F}$. If the limit of intersection of C_t and $C_{t'}$ as $t' \to t$ is (x, y), then (x, y) is a member of the limiting position envelope. Therefore, at the enveloping points, there is no change in $F(x, y, t)$ with respect to t. It then follows that all points on an enveloping curve satisfy $\frac{\partial F}{\partial t}(x, y, t) = 0$.

Through the combination of these two definitions, we arrive at the envelope algorithm.

Algorithm. *Let $\mathcal{F}$ be a family of curves such that each $C_t \in \mathcal{F}$ is given by $F(x, y, t) = 0$ for some fixed t, where C_t is smooth, and parameter t lies in an open interval; and let E be an envelope of $\mathcal{F}$. The following steps (if possible), will yield a closed form expression for E.*

1. *Evaluate* $F(x, y, t) = 0$ *for* t *in terms of* x *and* y.
2. *Evaluate* $\frac{\partial F}{\partial t}(x, y, t) = 0$ *for* t *in terms of* x *and* y.
3. *Combine the equations given by (1) and (2) via the elimination of* t.

This algorithm will be essential to the formation of a closed form equation of the curve found in Work No. 13 by Emma Kunz, as well as the solution to the ladder problem.

Two Proofs of the Ladder Problem

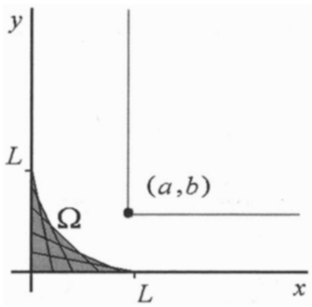

Figure 1: Ladder Problem[3]

The ladder problem asks to find the longest possible line-segment (the ladder) that can move freely around the corner (Fig.1). We imagine a vertical ladder on the wall, which is then gently kicked on the bottom, allowing it to slide across the floor while maintaining contact with the wall. This falling ladder traces an envelope, and the longest possible ladder is the one whose envelope intersects the point (a,b). Thus, our re-framed problem is to find a closed formula for the envelope of the falling ladder. We present 2 solutions, one using the algorithm and another using geometry.

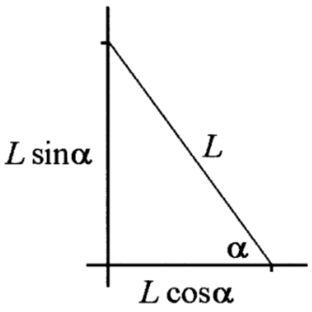

Figure 2: Diagram for First Solution

Solution 1. The positions of the ladder can be characterized by linear equations parameterized by the angle α (Fig 2) restricted to the interval $[0, \pi/2]$. Then the equation of the line is

$$\frac{x}{\cos\alpha} + \frac{y}{\sin\alpha} - L = 0 \tag{1}$$

where L is the length of the ladder. We differentiate with respect to α to get $\frac{x\sin\alpha}{\cos^2\alpha} - \frac{y\cos\alpha}{\sin^2\alpha} = 0$. We can rearrange to get: $\tan\alpha = \frac{y^{1/3}}{x^{1/3}}$, so

$$\cos\alpha = \frac{x^{1/3}}{\sqrt{x^{2/3}+y^{2/3}}} \text{ and } \sin\alpha = \frac{y^{1/3}}{\sqrt{x^{2/3}+y^{2/3}}}. \tag{2}$$

We substitute these results into equation (1) to eliminate α and arrive at $x^{2/3}\sqrt{x^{2/3}+y^{2/3}} + y^{2/3}\sqrt{x^{2/3}+y^{2/3}} - L = 0$. We can further simplify and obtain our solution $(x^{2/3}+y^{2/3})^{3/2} = L$. If we want a parametric form for our envelope, we can cube root both sides and substitute into (2) to get $x = L\cos^3\alpha$ and $y = L\sin^3\alpha$.

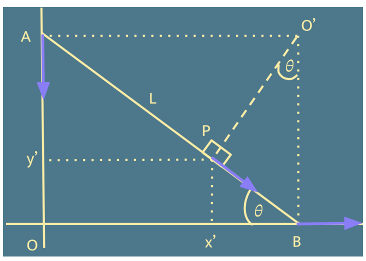

Figure 3: Picture Proof for Second Solution

Solution 2. Our next solution only requires basic geometry and an intuitive assumption that is verified by physics. The assumption is that the point P (Fig.3) on the ladder is the point of tangency with our desired envelope such that line O'P is perpendicular to the ladder. Then, by similar triangles, $\angle PO'B = \theta$. Then we have $|PB| = L\sin^2\theta$, $y = |X'P| = L\sin^3\theta$, $x = |OB| - |X'B| = L\cos\theta(1-\sin^2\theta)$, so $x = L\cos^3\theta$. We arrived at the same parameterization as our previous solution. If we want an equation as well, we simply solve for $\cos\theta$ and $\sin\theta$ and then use $\cos^2\theta+\sin^2\theta = 1$. This yields our equation $(\frac{x}{L})^{2/3} + (\frac{y}{L})^{2/3} = 1$, which is equivalent to our first solution.

Now to understand why our assumption was correct, recall that our problem is motivated by a physical ladder falling. At every instant, each point on the ladder has a certain velocity. Since the ladder is tangent to the envelope (by definition) the point of tangency must have a velocity directly down the ladder (otherwise the ladder would be moving into or out of the envelope). To find the point that has velocity down the ladder, we use the physics concept of instantaneous center of rotation. Based on the velocities of point A and B, we know the ladder is "rotating" about point O'. Thus, dropping a perpendicular onto the ladder reveals that point P does have the appropriate velocity, validating our assumption.

A Parabola in Plain Sight

Where the envelope of the ladder problem can be resolved more elegantly using geometry, other envelopes are better suited to the envelope algorithm. One such example would be the derivation of the envelope found in Work No. 13 by Emma Kunz (Figure 4).

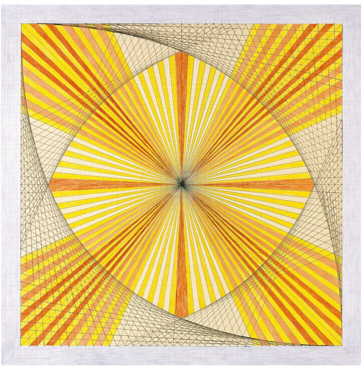

Figure 4: Work No. 13, Emma Kunz

Here, it can clearly be seen that the diagonal lines near the edges of this artwork form the illusion of four similar curves, each spanning a different quadrant. A closed form expression for these curves can be derived for each boundary curve through the application of the envelope algorithm. We will show our derivation of these enveloping curves as well as outline the general process for applying the envelope algorithm.

The first step in this derivation is to define $\mathcal{F}$, the family of curves. Considering first the family of curves which mostly lie in quadrant I, we may first observe that there are 40 lines in this family, which gradually change in slope from the horizontal line at $y = 20$ to the vertical line at $x = 20$, taking the origin to be located at the center and the width of the painting to span $[-20, 20]$. It then follows naturally that this family of curves can be defined by the sets of solutions (x, y) of $y = \frac{t}{t-40}(x - t + 20) + 20$, for each $t \in \mathbb{N}$, where t is a parameter such that $0 \leq t \leq 40$. Rearranging variables, we find family $\mathcal{F}_I = \{C_t : 0 < t < 40\}$ where each curve C_t is defined as:

$$C_t = \{(x, y) : t^2 - tx + ty - 40t - 40y + 800 = 0\}$$

Importantly, we extend $t \in (0, 40)$ to create a continuous, and differentiable family of curves with an equivalent envelope to that of the discrete curves found in the artwork. We can then begin to apply the envelope algorithm.

Given our quadratic condition $F(x, y, t) = t^2 - tx + ty - 40t - 40y + 800$, we can solve explicitly for t in terms of x and y as $t = \frac{1}{2}\left(\pm\sqrt{x^2 - 2xy + 80x + y^2 + 80y - 1600} + x - y + 40\right)$. Next, we solve for $F_t(x, y, t) = 2t - x + y - 40$, and solve for t in terms of x and y as $t = \frac{1}{2}(x - y + 40)$. Finally, solving both equations at the same fixed parameter t, we can simplify the curve

$$\frac{1}{2}\left(\pm\sqrt{x^2-2xy+80x+y^2+80y-1600}+x-y+40\right)=\frac{1}{2}(x-y+40)$$

which is the parabola E_I where $E_I = \{(x,y) : x^2+80x-2xy+80y+y^2-1600=0\}$.

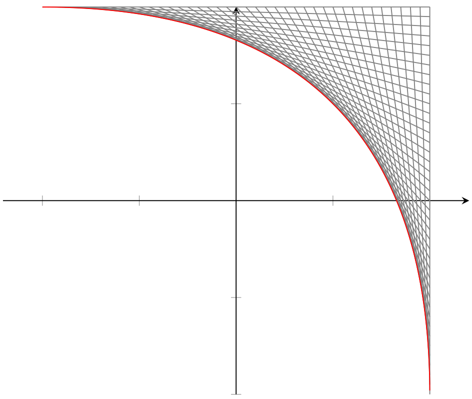

Figure 5: Envelope in Quadrant I

Graphically, we can verify that this parabola does in fact form an envelope on the family of curves parameterized in quadrant I, as seen in Figure 5. The tangent curves $C_t \in \mathcal{F}$ are plotted in grey, and the enveloping parabola E_I is plotted at the boundary in red. Note that the curves C_t plotted are the curves C_t where $t \in 0,\ldots,40$, rather than the continuous family of curves C_t where $t \in (0,40)$. To validate the calculus used in the envelope algorithm, the family of curves must be parameterized on an open interval. This is because $\frac{\partial F}{\partial t}$ only exists for a continuous and differentiable parameterization. In Kunz's artwork, only curves parameterized by an integer are drawn, and Figure 5 is meant to recreate closely her work. Figure 6 in the Appendix showcases a full recreation of all four families of curves and their boundary envelopes.

Following the same procedure, we can define families of curves for each quadrant:

$$\begin{aligned}
\mathcal{F}_I &= \{C_t : 0<t<40\} \text{ where } C_t = \{(x,y) : t^2-tx+ty-40t-40y+800=0\}\\
\mathcal{F}_{II} &= \{C_t : 0<t<40\} \text{ where } C_t = \{(x,y) : t^2+tx+ty-40t-40y+800=0\}\\
\mathcal{F}_{III} &= \{C_t : 0<t<40\} \text{ where } C_t = \{(x,y) : t^2+tx-ty-40t+40y+800=0\}\\
\mathcal{F}_{IV} &= \{C_t : 0<t<40\} \text{ where } C_t = \{(x,y) : t^2-tx-ty-40t+40y+800=0\}
\end{aligned}$$

And solve explicitly for the parabolas which form an envelope on them:

$$\begin{aligned}
E_I &= \{(x,y) : x^2+80x-2xy+80y+y^2-1600=0\}\\
E_{II} &= \{(x,y) : x^2-80x+2xy+80y+y^2-1600=0\}\\
E_{III} &= \{(x,y) : x^2-80x-2xy-80y+y^2-1600=0\}\\
E_{IV} &= \{(x,y) : x^2+80x+2xy-80y+y^2-1600=0\}
\end{aligned}$$

Conclusion

In solving for these envelopes, I am reminded of a quote by Michael Atiyah:

> Algebra is the offer made by the devil to the mathematician. The devil says: I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine.

In this way, the beauty of mathematics is often obscured by the treachery of algebra. No clearer is this seen than in the difference between our two proofs of the Ladder Problem. In the solution given by Kalman,[3] trigonometry wrangles the algebra until it is reduced to a series of exponents. It is not immediately obvious why the resulting equation is true, and it is rather miraculous that the algebra was navigated in the first place. But, alas, the algebra works its magic and determines the curve to be $(x^{2/3} + y^{2/3})^{3/2} = L$.

Contrast this now with the geometric picture proof. Through the drawing of a few clever azimuths, we can create similar triangles and see how naturally the solution falls from the trigonometry inherent to the problem. Clearly, this second solution forms a more intuitive understanding of the Ladder Problem than the raw algebra in the first solution.

But more than intuitive, the geometric solution to the Ladder Problem is beautiful. So clearly does the asteroid curve follow from the geometry that one cannot fail to appreciate the wonder therein. Similarly, the beauty inherent to art is self-evident, and yet when the formulas are abstracted from the painting of Emma Kunz, the beauty seems to disappear alongside the geometry. The derivation is messy vector calculus, which takes striking families of curves and translates them into dull multi-variable functions. Then, from algebra that would only excite the most passionate of high-school teachers, a parabola appears as if it were magic. Yet, unlike the odd letters which denote strange sets, the parabola itself is tangible, It can be graphed, it can be shown. And when the family of curves is graphed, and the envelope is drawn to fill the imaginary limits of intersections our minds have already filled, the image is reignited from the beauty lost from the algebraic steps.

We made this sacrifice long ago to trade geometry for algebra, and in doing so hid the beauty of mathematics behind functions and *math*. Horrible, horrible *math*. However, envelopes provide the rare occasion where the geometry becomes unobscured and rejuvenated with beauty. We find again what we have lost, what we have sacrificed. We can show these things to a layman and if nothing else, communicate the beauty inherent to mathematics. More than interest, more than curiosity, this is the importance of studying mathematical art. And we have found that envelopes are a wondrous tool to extract geometry from algebra and showcase math that anyone may appreciate.

References

[1] Kelly Bickel, Pamela Gorkin, and Trung Tran. Applications of envelopes. *Complex Analysis and its Synergies*, 6(1):2, Jan 2020.

[2] R. Courant. *Differential and Integral Calculus, Volume 2*. Wiley Classics Library. Wiley, 2011.

[3] Dan Kalman. Solving the ladder problem on the back of an envelope. *Mathematics Magazine*, 80(3):163–182, 2007.

Appendix

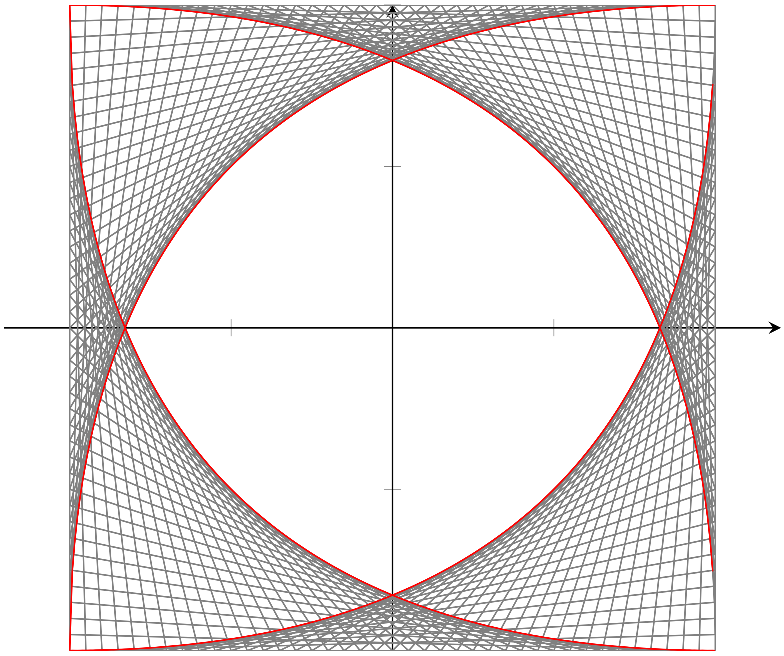

Figure 6: Full Plot of Families of Curves and their Envelopes